Keypad Geometry and Divisibility of Numbers

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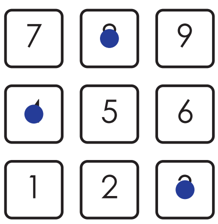
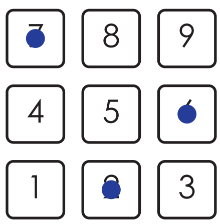
In this article we show how students can form familiar geometric figures on the calculator keypad and generate numbers that are all divisible by a common number. Students are intrigued by the results and want to know "why it works". The activities can be presented and students given an extended amount of time to think about them. As students can easily generate their own examples, they can continue to hypothesise why the property holds as the semester moves on. The proofs of the more complicated figures should be given only to those who are truly interested.

First we generate numbers divisible by 3. A three-digit number for which no digit is in the same column as any other will be divisible by 3. Choose any digit to start; the next digit cannot be in the same column as the first so there are six choices left. Choosing the second digit eliminates another column and leaves three choices for the third digit. We illustrate this with two examples:

$483 = 3 \times 161$



$672 = 3 \times 224$



As we have nine choices to start with, six choices for the second digit and then three for the third, we can generate 162 numbers in this way. A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

On the keypad, the final column has 3 numbers all of which are divisible by 3 ($9 = 3 \times 3$, $6 = 3 \times 2$, $3 = 3 \times 1$). The numbers in the second column each have a remainder of 2 when divided by 3 ($8 = 2 \times 3 + 2$, $5 = 1 \times 3 + 2$, $2 = 0 \times 3 + 2$) and the numbers in the first column each have a remainder of 1 when divided by 3 ($7 = 2 \times 3 + 1$, $4 = 3 \times 1 + 1$, $1 = 0 \times 3 + 1$). By forming the number as we did, choosing three digits where no digit is in the same column as any other, we are guaranteed to have one digit from column one, one from column 2 and the other from column 3. Adding the digits will result in a multiple of 3 as the remainders add to 3. Students may ask why is a number divisible by 3 if the sum of its digits is divisible by 3. Below is a proof of this for our 3 digit numbers.

The notation $[abc]$ denotes the number $a \times 10^2 + b \times 10 + c$

$$\begin{aligned}
[abc] &= a \times 102 + b \times 10 + c \\
&= a(99 + 1) + b(9 + 1) + c \\
&= a(99) + a + b(9) + \mathrm{b} + c \\
&= a(99) + b(9) + a + b + c
\end{aligned}$$

The above shows that $[abc] = a(99) + b(9) + a + b + c$ and so $[abc]$ is divisible by 3 if and only if $a + b + c$ is divisible by 3. We showed above for all our examples the sum of the digits, $a + b + c$, was divisible by 3 so our numbers are all divisible by 3. This argument can be easily generalised to prove that any natural number is divisible by 3 or 9 if and only if the sum of the digits is divisible by 3 or 9 respectively.

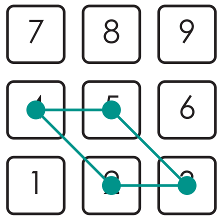
When we mark one digit from each column with a dot and connect the dots to look at the figure formed we will always get a triangle or a straight line. We have shown for the triangle you can start at any vertex, go clockwise or counter-clockwise around the vertices, writing down the three digit number that results and you get a number divisible by 3. Likewise for the straight line you can put the 3 digits covered in any order and again you have a number divisible by 3. Although divisibility by 3 holds for a three-digit number arising along any line (vertical, diagonal, or horizontal), it will not hold for all triangles. If two of the vertices of the triangle come from the same column, the resulting 3 digit number will not be divisible by 3. This is because it is impossible for the sum of the digits to be divisible by 3 if two of the vertices come from the same column. Students can form challenge triangles and the teacher can immediately see if the 3 digit numbers resulting from the vertices will be divisible by 3 or not.

In contrast to triangles, all parallelograms which can be formed using the keypad yield four-digit numbers divisible by the same number (11, 111, or 1111). Consider the number keypad below:

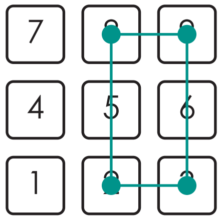
7	8	9
4	5	6
1	2	3

Using the numerals on the keypad as vertices, form any parallelogram; start at any vertex and go in a clockwise or counterclockwise direction writing down the four-digit number obtained from the single digit numbers at the corners. The result will always be divisible by 11. We illustrate this with two examples:

$3245 = 11 \times 295$



$9823 = 11 \times 893$



As before, we use the notation $[abcd]$ to denote the number $a \times 10^3 + b \times 10^2 + c \times 10 + d$.

If the number you have written down is of the form $[abcd]$, you can see $b - a = c - d$. If we take two parallel line segments of equal length on a keypad with endpoints at keys x, y for one and w, z for the other, then $|\ x - y\ | = |\ w - z\ |$.The displacement from one endpoint to the other can be calculated by first moving horizontally and then vertically. For both line segments, it is clear the magnitude of the displacement is the same. In this case, we can see that $b - a$ and $c - d$ will have the same sign, so we can eliminate the absolute values. Now $b - a = c - d$ implies $a + c = b + d$ which implies the number is divisible by 11. The fact that the sum of the digits in odd positions subtracted from the sum of the digits in even positions is zero is a sufficient condition for divisibility by 11.

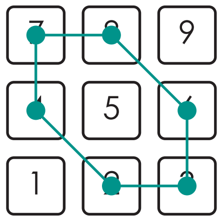
To see why this works, we expand $[abcd]$ in terms of its place value. We then add and subtract "a," "b," and "c" terms and group like terms to find expressions all divisible by 11:

$$\begin{aligned}[abcd] &= a \times 10^3 + b \times 10^2 + c \times 10 + d \\ &= a \times 10^3 + a + b \times 10^2 - b + c \times 10 + c - a + b - c + d \\ &= a \times (1001) + b \times (99) + c \times (11) + ((b - a) - (c - d)) \\ &= a \times (91 \times 11) + b \times (9 \times 11) + c \times (11) + 0\end{aligned}$$

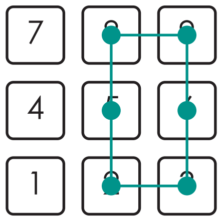
This argument can be generalised to prove that a number is divisible by 11 if and only if the sum of the digits in odd positions subtracted from the sum of the digits in even positions is divisible by 11.

For a bigger challenge consider a regular hexagon or rectangle going through six keys. Start at any key and go in a clockwise or counterclockwise direction writing down the six digit number obtained from the single digit numbers along the edges. This number will always be divisible by 111. Again we illustrate with two examples.

$742\,368 = 111 \times 6688$



$523\,698 = 111 \times 4718$



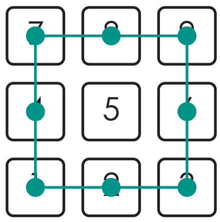
Here if the number obtained from the hexagon is of the form $[abcdef]$, the parallel sides of equal length imply $b - a = d - e$, $c - b = e - f$,. so $a + d = b + e = c + f$. This time we use this as well as add and subtract well chosen "a," "b," and "c" terms. We group like terms to find expressions all divisible by 111:

$$\begin{aligned} \text{Now}\quad [abcdef] &= a \times 10^5 + b \times 10^4 + c \times 10^3 + d \times 10^2 + e \times 10 + f \\ &= a \times 10^5 + b \times 10^4 + c \times 10^3 + d \times 10^5 + e \times 10^4 + f \times 10^3 \\ &\quad - d \times 10^5 - e \times 10^4 - f \times 10^3 + d \times 10^2 + e \times 10 + f \\ &= (a + d)10^3(111) - [def] \times 999 \end{aligned}$$

and so is divisible by 111.

Next consider the square going through all the digits but 5. Start at any number along a side, write down the 8 digit number that results from going clockwise or counterclockwise around the square. This number will always be divisible by 1111. For example:

63 214 789 = (1111)(56 899).



Suppose the number is of the form $[abcdefgh]$. As above, $b - a = e - f$, $c - b = f - g$, $d - c = g - h$ so $a + e = b + f = c + g = d + h$. The process is now familiar.

$$\begin{aligned} [abcdefgh] &= a \times 10^7 + b \times 10^6 + c \times 10^5 + d \times 10^4 + e \times 10^3 + f \times 10^2 \\ &\quad + g \times 10^1 + h \\ &= a \times 10^7 + b \times 10^6 + c \times 10^5 + d \times 10^4 + e \times 10^7 + f \times 10^6 \\ &\quad + g \times 10^5 + h \times 10^4 - e \times 10^7 - f \times 10^6 - g \times 10^5 \\ &\quad - h \times 10^4 + e \times 10^3 + f \times 10^2 + g \times 10^1 + h \\ &= a \times 10^7 + b \times 10^6 + c \times 10^5 + d \times 10^4 + e \times 10^7 + f \times 10^6 \\ &\quad + g \times 10^5 + h \times 10^4 - [efgh] \times 9999 \\ &= (a + e)10^4(1111) - [efgh] \times 9999 \end{aligned}$$

and so is divisible by 1111.

The following website gives a test and proof of test for divisibility by strings of ones: www.ken.duisenberg.com/potw/archive/arch97/971114sol.html. The idea is to see if a number is divisible by a string 11…1 (k ones) break up the number into strings of length k starting from the units digit and adding zeros on the left to the final string if need be. Add all these strings together and if this result is divisible by 11…1 (k ones) the original number is also. All our polygonal examples are easily seen to fit the criterion. A colleague Dan Kalman introduced us to divisibility properties of symmetric figures on the calculator. He, in turn, credited Michael Wage and Timothy N. Kaiser with first discovering the idea in the 1980s.

From Helen Prochazka's

Scrapbook

If you ask mathematicians what they do, you always get the same answer. They think. They think about difficult and unusual problems. They do not think about ordinary problems: they just write down the answers.

M. Egrafov, mathematician